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DETERMINANTS BY USING GENERATING FUNCTIONS

MASRESHAW WALLE ABATE

Department of Mathematics, Dilla University, Dilla, Ethiopia

ABSTRACT

As we know, let alone to find the determinant of infinite matrix, it is difficult to find the determinant of some n x n matrixes by the usual methods like, the cofactor method and Crammer's rule. But now we will show how to find the determinant of some n x n matrices and how to find the determinant of some infinite matrix by using Generating Functions. In this paper we will consider matrices having 1's on the supper diagonal, 0's on the upper and identical entries on each diagonal below the supper diagonal. Here we will try how to obtain the determinant of n x n upper left corner sub matrix of a given **infinite matrix** by introducing Generating functions of some sequences and how to get a sequence by calculating the determinant of n x n upper left corner sub matrix of infinite matrix. We will also check the correctness of the determinant by using Numerical method

KEYWORDS: Infinite Matrix; Determinant of Matrices; Generating Functions; Sequences; Sub Matrix

1. INTRODUCTION

To understand the whole work, it is better to know about a matrix, determinants, Generating functions and some sequences. So we will discuss these terms before the actual work.

What are Generating Functions?

One of the main tasks in combinatorics is to develop tools for counting. Perhaps, one of the most powerful tools frequently used in counting is the notion of Generating functions. Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series. Generating functions can be used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints, and the number of ways to make change for a dollar using coins of different denominations (Discrete Mathematics and Its Applications, Seventh Edition, Kenneth H. Rosen Monmouth University (and formerly AT&T Laboratories page 537). In mathematics a Generating function is a formal power series whose coefficients encode information about a sequence $\{a_n\}$ that is indexed by the natural number n. Generating functions can be used to solve determinants of some nxn and then an infinite matrix by relating the terms of the sequence for which we get a generating function to the determinant of an upper left corner nxn matrix of an infinite matrix.. Even though there are various types of Generating functions, in this paper, we introduce the idea of ordinary generating functions **(OGF)** and look at some ways to manipulate them. Even though we only consider the ordinary Generating functions we will also define Exponential Generating functions **(EGF)** in this paper.

We begin with the definition of the generating function for a sequence.

Definition

Ordinary generating function (OGF) supposes we are given a sequence a_0, a_1, the ordinary generating function (also called OGF) associated with this sequence is the function whose value at x is$\sum_{i=0}^{\infty} a_i x^i$. The sequence a0, a_1, is called the coefficients of the generating function. People often drop “ordinary” and call this the generating function for the sequence. This is also called a “power series” because it is the sum of a series whose terms involve powers of x (CHAPTER 10 Ordinary Generating Functions)

I.e Let $(a_n) = (a_1, a_2, \dots, a_r, \dots)$ is a sequence of numbers. The Generating function for the sequence (a_n) is defined to be the power series:-

i) $A(x) = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$ for Ordinary generating function (OGF)

ii) $A(X) = \sum_{r=0}^{\infty} a_r \frac{x^r}{r!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots \quad a_r \frac{x^r}{r!} \dots$ for Exponential Generating function (EGF).

Some Examples of Generating Functions of Some Sequences

i) $\langle 1,1,1,\dots \rangle \leftrightarrow 1 + x + x^2 + x^3 + \dots = \sum_{n-0}^{\infty} x^n = \frac{1}{1-x}$ is the ordinary Generating function

ii) $\langle 1,1,\frac{1}{2!},\frac{1}{3!},\frac{1}{4!},\dots \rangle \leftrightarrow \sum_{r=0}^{\infty} \frac{x^r}{r!} = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ is its exponential Generating function.

iii) $\langle 1,2,3,4,\dots \rangle \leftrightarrow 1+2x+3x^2+4x^3+\dots = \frac{1}{(1-x)^2}$ is the generating function for counting numbers .

iv) The generating function for the sequence $(1, k, k^2, k^3, \dots)$, where k is an ordinary constant is

$1+kx+k^2x^2+k^3x^3+\dots = \frac{1}{1-kx}$

v) $\langle 1,-1,1,-1,\dots \rangle \leftrightarrow 1 - x + x^2 - x^3 + \dots = \frac{1}{1-(-x)} = \frac{1}{1+x}$

vi) $\langle 1, a, a^2, a^3, \dots \rangle \leftrightarrow 1 + ax + a^2x^2 + \dots = \frac{1}{1-ax}$

vii) $\langle 1,0,1,0,1,\dots \rangle \leftrightarrow 1 + x^2 + x^4 + x^6 + \dots = \frac{1}{1-x^2}$

Two Generating functions A(x) and B(x) for the sequence (a_r) and (b_r), respectively are considered equal (written A(x) = B(x)) $\Leftrightarrow a_i = b_i \ \forall_i \in N$. In considering the summation in a Generating function, we may assume that x has been chosen such that the series converges. In fact, we do not have to concern ourselves so much with the questions of convergence of the series, since we are only interested in the coefficients. Ivan Niven [N] gave an excellent account of the theory of formal power series that allow us to ignore questions of convergence, so that we can add and multiply formal power series term by term like polynomials.

The technique of generating function is useful in the study of at least one sequence, that is, the binomial coefficients.

1.2 OPERATIONS ON GENERATING FUNCTIONS

Let $A(x) = a_0+a_1 x+a_2 x^2 + a_3 x^3+\ldots$ and $B(x) = b_0+b_1 x+b_2 x^2 + b_3 x^3+\ldots$ be the generating functions for the sequences (a_r) and (b_r) respectively, then

1.2.1 Constant /Scaling/ Rule

$\langle ca_0, ca_1, ca_2, ca_{3\ldots} ca_{n\ldots} \rangle \leftrightarrow cA(x)$.

Proof

$\langle ca_0, ca_1, ca_2, ca_{3\ldots} \rangle \leftrightarrow ca_0 + ca_1x+ca_2 x^2+ca_3x^3 +\ldots$

$=c\,(a_0+a_1x+a_2x^2+a_3 x^3+\ldots \rangle = cA(x)$

- ***E.g.*** If $\langle 1,2,3,4,\ldots \rangle \leftrightarrow 1+2x+3x^2+ 4x^3+\ldots = \dfrac{1}{(1-x)^2}$ and c=2, we have

$\frac{2}{(1-x)^2} = 2 + 4x + 6x^2 + 8x^3 + \ldots \leftrightarrow (2,4,6,8,10,\ldots)$

1.2.2 Addition Rule

A(x)+B(x) is the generating function for the sequence (c_r) where $c_r= a_r+ b_r$,

$r = 0,1,2,3\ldots$

Proof

$A(x)+B(x) = (a_0+a_1x+a_2 x^2 + a_3 x^3+\ldots) + (b_0+b_1x+b_2x^2 + b_3 x^3+\ldots)$

$= (a_0+ b_0) + (a_1 + b_1)x+(a_2+ b_2)x^2 +(a_3+ b_3.)x^3+\ldots$

$= c_0+ c_1x + c_2x^2+ c_3x^3+\ldots$

Where $c_r= a_r+ b_r$, $r = 0,1,2,3\ldots$

eg. if $A(x) = \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots \leftrightarrow \langle 1,2,4,8,\ldots \rangle$

$B(x) = \frac{3}{1-3x} = 3\left(\frac{3}{1-3x}\right) = 3(1 + 3x + 9x^2 + 27x^3 + \cdots) = 3 + 9x + 27x^2 + 81x^3 + \cdots \leftrightarrow \langle 3,9,27,\ldots \rangle$

Then $A(x) + B(x) = \dfrac{1}{1-2x} + \dfrac{3}{1-3x} = \dfrac{4-9x}{1-5x+6x^2} \leftrightarrow \langle 4, 11, 31\ldots \rangle$

1.2.3. Product Rule

$A(x)\times B(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \ldots$ is the generating function for the sequence (c_r), where

$c_r = a_0b_0 + a_1b_{r-1} + \ldots a_{r-1}b_1 + a_rb_0$, r = 0,1,2,3,...

Proof

To evaluate the product $A(x) \times B(x)$ let us use the following table.

Table 1 Product Table

×	b_0	b_1x	b_2x^2	$b_3x^3\ldots$
a_0	a_0b_0 →	a_0b_1x	$a_0b_2x^2$ →	$a_0b_3x^3\ldots$
a_1x	a_1b_0x	$a_1b_1x^2$	$a_1b_2x^3$	$a_1b_3x^4$
a_2x^2	$a_2b_0x^2$	$a_2b_1x^3$	$a_2b_2x^4$	$a_2b_3x^5\ldots$
a_3x^3	$a_3b_0x^3$	$a_3b_1x^4$	$a_3b_2x^5$	$a_3b_3x^6\ldots$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

If we follow the arrow, we get the required product

E.g. If $A(x) \leftrightarrow \langle 1,2,2,2, \rangle \leftrightarrow \frac{1+x}{1-x}$ and $B(x) \leftrightarrow \langle 1, 1, 1,\ldots \rangle \leftrightarrow \frac{1}{1-x}$ then

$$A(x).\,B(x) = \left(\frac{1+x}{1-x}\right)\left(\frac{1}{1-x}\right) = \frac{1+x}{(1-x)^2} \leftrightarrow \langle 1,3,5,7,9\ldots \rangle$$

Using the product rule we have the following:

$(1-x)\,A(x)$ is the generating function for the sequence (c_r) where

$c_0 = a_0$ and $c_r = a_r - a_{r-1}$ for all $r \geq 1$ and

$\frac{A(x)}{1-x}$ Is the generating function for the sequence (c_r) where?

$c_r = a_0 + a_1 + a_2 + \ldots + a_r$ for all r.

Remark

since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots..$ and

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^n}{n!} + \ldots..$$

Then we have $\frac{e^{x}+e^{-x}}{2}=1+\frac{x^{2}}{2!}+\frac{x^{4}}{4!}+\ldots=\cos x$ and

$$\frac{e^{x}-e^{-x}}{2} = x+\frac{x^{3}}{3!}+\frac{x^{5}}{5!}+\ldots=\sin x$$

1.3.4. SHIFT RIGHT RULE

Suppose $A(x) \leftrightarrow \langle a_0, a_1, a_2 .. \rangle$

$\Rightarrow xA(x) \leftrightarrow \langle 0, a_0, a_1, a_2, \ldots \rangle$

$\Rightarrow x^2A(x) \leftrightarrow \langle 0, 0, a_0, a_1, a_2, \ldots \rangle$

$x^mA(x) \leftrightarrow \langle \underbrace{0,0,0 \ldots, 0,}_{\text{m-zero}} a_0, a_1, a_2, \ldots \rangle$

1.2.5. THE DERIVATIVE RULE

If $\langle a_0, a_1, a_2 .. \rangle \leftrightarrow A(x)$, then $\langle a_1, 2a_2, 3a_3, \ldots \rangle \leftrightarrow A'(x)$

Proof

$\langle a_0, a_1, a_2, \ldots \rangle \leftrightarrow A(x) = a_0 + a_1x + a_2 x^2 + a_3 x^3+\ldots$

$$\Rightarrow \frac{d}{dx} A(x) = a_1+2a_2 x + 3a_3x^2+\ldots = A'(x) \leftrightarrow \langle a_1, 2a_2, 3a_3^{2,}\ldots \rangle$$

- E.g. $\langle 1,1,1,\ldots \rangle \leftrightarrow \frac{1}{1-x} = 1 + x + x^2 + x^3+\ldots= A(x)$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1+2x + 3x^2 +\ldots \leftrightarrow \langle 1, 2, 3, 4\ldots \rangle = A'(x)$$

And $x A'(x) \leftrightarrow \langle 0, 1, 2, 3\ldots \rangle \leftrightarrow \frac{x}{(1-x)^2}$

Hence $[xA`(x)]` = \left(\frac{x}{(1-x)^2}\right)' = \frac{(1+x)}{(1+x)^3} \leftrightarrow \langle 1, 4, 9, 16,\ldots \rangle$ (square number Sequence)

Note: $\lfloor x^n \rfloor$ Given a generating function A(x) we use $\lfloor x^n \rfloor$ A(x) to denote a_n, the coefficient of x^n (270 Chapter 10 Ordinary Generating Functions)

MATRIX AND DETERMINANTS

Definition

A matrix is a rectangular array of mn quanties $a_{ij}\begin{pmatrix} i=1,2,3,....,m\ \& \\ j=1,2,3,...,n \end{pmatrix}$ in m - rows and n- columns. It is called an m $\times$n matrix or a matrix of order m $\times$ n and read as m by n matrix. The numbers a_{ij} are called the elements (constituents or coordinates or entries) of the matrix and we will denote the matrix by $\{a_{ij}\}$ or A. The suffix ij of an element a_{ij} indicates that it occurs in the i^{th} row and j^{th} column. when n=m we call this a square matrix. For a square matrix nxn, if n$\rightarrow\infty$ then the matrix is called an infinite matrix.

In Explicit form $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & a_{3n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & a_{nn} \end{pmatrix}$ rows →

Columns

NOTE

1) If A = $\{a_{ij}\}_{m \times n}$ and B= $\{b_{ij}\}_{n \times p}$, and AB =C, then C = $\{c_{ij}\}_{m \times p}$

Where cij = $\sum_{k=1}^{n} a_{ij} b_{kj}$

2) For any matrix

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & a_{3n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & a_{n3} & . & . & a_{nn} \end{pmatrix}$$

The supper diagonal

The main diagonal

Sub diagonal

MATRIX MUITIPLLCATION

Two matrixes A and B are conformable for the product AB when the number of columns in A is equal to the number of rows in B. If A is an m x n matrix and B is an n x p matrix then their product AB is defined as m x p matrix whose (ij) th element is obtained by multiplying the element of the i^{th} row of A in the corresponding elements of the j^{th} column of B and summing the products so obtained. So the (ij) th element of the product AB is the inner product of the ith row of A and the j th column of B.

DETERMINANTS

Definition

Determinant of a matrix A is a specific real number assigned to A It is denoted by det (A) or $|A|$ Or for n $\geq$ 1 the determinant of an n x n matrix A= (a_{ij}) along the first row is the sum of n-terms of the form $\pm$ a_{ij} det A_{ij} with plus and minus signs alternating where the entries $a_{11}, a_{12}, a_{13}...a_{1n}$ are from the first row of A.

In symbols, det A = a_{11} det A_{11} - a_{12} det A_{12} +... + $(-1)^{n+1}$ a_{1n} det A_{1n} = $\sum_{j=1}^{n}$ $(-1)^{1+j}$ a_{1j} A_{1j} and det A_{1j} is the determinant of the sub matrix which is obtained by removing the 1st row and the jth column.

Actually det A_{1j} is called the minor of a_{1j} and $(-1)^{1+j}$ a_{1j} det A_{1j} is called the cofactor of a_{ij}.

CRAMER'S RULE

Let A be an invertible n x n matrix. For any b in R^n the unique solution x of

Ax = b has entries given by $x_i = \frac{detA_i(b)}{detA}$ where i = 1,2, 3,... n and A_i (b) is the matrix obtained from A by replacing column i by the vector b.

Proof

Denote the column of A by $a_1, a_2, a_3,... a_n$ and the column of the n x n identity matrix I by $e_1, e_2, e_3,... e_n$.

If Ax =b then the definition of matrix multiplication shows that

$A I_i (x) = A [e_1, e_2, e_3,... x... e_n] = [Ae_1, Ae_2... Ax... Ae_n]$

$= [a_1, a_2 ... b... a_n] = A_i (b)$

by the multiplicative property of determinants

(det A) det I_i (x) = det A_i (b)

=> (detA) X_i = det A_i (b)

=> $X_i = \frac{detA_i(b)}{detA}$

Determinant of a matrix can be obtained by the cofactor method or by using the Cramer's rule. But now we are Interested to show how to find the determinant of several matrices by using Generating functions.

The matrices whose determinants we will be evaluating have all 1's on the super diagonal, 0's above the supper diagonal ; and identical entries on each diagonal below the supper diagonal, perhaps with the exception of the first column,

2. DESCRIPTION OF THE METHOD

In this topic we will see how to get a sequence from the given matrix by calculating the determinant of each upper left nxn square matrices of the given matrix. All matrices in this section will have 1's on the supper,0`s above and identical entries on each diagonal below, perhaps with the exception of the first column. Hence we will equate the n^{th} term of the sequence with determinant of each upper left nxn square matrices of the given matrix. We begin with a typical example as follows

Example 1: Suppose

$$A=\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ -2 & 2 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ 2 & 0 & 2 & 1 & 0 & 0 & 0 & . & . & . \\ -2 & 0 & 0 & 2 & 1 & 0 & 0 & . & . & . \\ 2 & 0 & 0 & 0 & 2 & 1 & 0 & . & . & . \\ -2 & 0 & 0 & 0 & 0 & 2 & 1 & . & . & . \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & . & . & . \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}. \quad (1)$$

be the given matrix with upper left square sub matrices $(2), \begin{pmatrix} 2 & 1 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ -2 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix}, \ldots$

Now we want to evaluate the upper left corner determinants as follows

$$|2| = 2, \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} = 6, \begin{vmatrix} 2 & 1 & 0 \\ -2 & 2 & 1 \\ 2 & 0 & 2 \end{vmatrix} = 14, \begin{vmatrix} 2 & 1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 2 & 0 & 2 & 1 \\ -2 & 0 & 0 & 2 \end{vmatrix} = 30, \ldots \ldots \ldots$$

where the n^{th} such determinant will be denoted by $D_{n,n} \geq 1$

One way to determine say D_5 is as follows, consider the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \\ 2 \end{pmatrix} \quad (2)$$

where the right hand side is the first column from the original matrix of (1) for n=5

By crammer's rule and properties of determinants, we have

$$a_5 = \frac{\begin{vmatrix} 1 & 0 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 & -2 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 0 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{vmatrix}}$$ = 62 In general by induction we have $a_n = (-1)^{n-1} D_n$. **(3)**

Now let us introduce the generating functions for the columns of (1) and rewrite it as (2) we get the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 2x & x & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 2x^2 & x^2 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 2x^3 & x^3 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 2x^4 & x^4 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 2x^5 & x^5 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} 2 \\ -2x \\ 2x^2 \\ -2x^3 \\ 2x^4 \\ -2x^5 \\ . \\ . \\ . \end{pmatrix} \quad \textbf{(4)}$$

Then the right hand side has a generating function

$$2 - 2x + 2x^2 - 2x^3 + \ldots = \frac{2}{1+x}$$

and except the elements on the main diagonal of the first matrix of left side of (4),

The first column has a generating function C(x) = 2x

The 2^{nd} column has a generating function xC(x) = $2x^2$

The 3^{rd} column has a generating function $x^2C(x) = 2x^3$

Letting $A(x) = a_1 + a_2 x + a_3 x^2 \ldots$ as the generating function for the sequence $\langle a_1, a_2, a_3, \ldots \rangle$ and summing on both sides of (4) we get

$$A(x) + a_1 C(x) + a_2 xC(x) + a_3 x^2 C(x) + \ldots = 2\left(\frac{2}{1+x}\right)$$

$$\Rightarrow A(x) + C(x)(a_1 + a_2 x + a_3 x^2 + \ldots) = \frac{2}{1+x}$$

$$\Rightarrow A(x) + A(x)\, C(x) = \frac{2}{1+x}$$

$$\Rightarrow A(x) = \frac{2}{1+x}(1+C(x)) = \left(\frac{1}{1+x}\right)\left(\frac{1}{1+2x}\right) = \frac{A}{1+2x} + \frac{B}{1+x} = \frac{4}{1+2x} - \frac{2}{1+x}$$

$\Rightarrow [x^n]\ A(x) = 4(-2)^n - 2(-1)^n = (-1)^n (4\times 2^n - 2) = (-1)^n (2^{n+2} - 2)$

$\Rightarrow a_{n+1} = (-1)^n (2^{n+2} - 2)$

$\Rightarrow a_n = (-1)^{n-1} (2^{n+1} - 2)$..... (5)

Equating equation (5) and (3) we have

$\mathbf{D_n = 2^{n+1} - 2 = 2(2^n - 1)} = \langle 2,6,14,30,62,126,....\rangle \forall n \geq 1$

Example 2:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & . & . & . \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & . & . & . \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & . & . & . \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & . & . & . \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}. \quad \textbf{(1)}$$

(1) is a given matrix with upper left square matrix $[1], \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \ldots$

Now we want to evaluate the determinant of the upper left corner matrix as follows

$$|1| = 1 = D_1, \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 = D_2, \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 3 = D_3, \begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{vmatrix} = 5 = D_4, \ldots \ldots \ldots$$

The n^{th} such determinant is denoted by D_n. One way to determine one of these say, D_5 is as follows. Consider the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \textbf{(2)}$$

Where the right hand side is the first column from the original matrix of (1) for n=5

By Cramer's rule and the properties of determinants, we have

$$a_5 = \frac{\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}} = 8 = D_5$$ In general by induction we have $a_n = (-1)^{n-1} D_n$ (3)

Now let us introduce a generating function for the column of (1) and rewrite it as (2). Then we get the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ x & x & 0 & 0 & 0 & 0 & . & . & . \\ 0 & x^2 & x^2 & 0 & 0 & 0 & . & . & . \\ x^3 & 0 & x^3 & x^3 & 0 & 0 & . & . & . \\ 0 & x^4 & 0 & x^4 & x^4 & 0 & . & . & . \\ x^5 & 0 & x^5 & 0 & x^5 & x^5 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} 1 \\ -x \\ x^2 \\ -x^3 \\ x^4 \\ -x^5 \\ . \\ . \\ . \end{pmatrix} \quad \textbf{(4)}$$

Then the right hand side of (4) has a generating function

$x + x^2 - x^3 + x^4 - x^5 + \dots = \frac{1}{1+x}$

And, except the elements on the main diagonal of the first matrix of left side of (4),

The first column has a generating function $C(x) = x + x^3 + x^5 + x^7 + \dots = \frac{x}{1-x^2}$

The 2nd column of has a generating function $x\,C(x) = x^2 + x^4 + x^6 + \dots$

The 3rd column has a generating function $x^2 C(x) = x^3 + x^5 + x^7 + \dots$

Letting $A(x) = a_1 + a_2 x + a_3 x^2 + \dots$ as the generating function for the sequence $\langle a_1, a_2, a_3 \dots \rangle$ and summing on both sides of (4) we get

$$A(x) + a_1 (x+x^3+x^5+\dots) + a_2 x (x+x^3+x^5+x^7\dots) + a_3 x^2 (x+x^3+x^5+x^7+\dots) = \frac{1}{1+x}$$

i.e. $$A(x) + a_1 \frac{1}{1-x^2} + a_2 x \frac{x}{1-x^2} + a_3 x^2 \frac{1}{1-x^2} + \dots = \frac{1}{1+x}$$

$\Rightarrow$ A (x) + A (x) $\dfrac{1}{1-x^2}$ = $\dfrac{1}{1+x}$

$\Rightarrow$ A (x) = $\frac{1-x^2}{(1+x)(1+x-x^2)}$ = $\frac{1-x}{1+x-x^2}$ = 1-2x+3x^2- 5x^3+8x^4+...

For n$\geq$1, $\lfloor x^n \rfloor A(x)$ a_n = (-1) $^{n-1}$ F_{n-1} **(5)**

Where F_n is the n th Fibonacci number. $\Rightarrow$Equating (5) and (3) we get

$D_n = F_n, \forall\, n > 1$

Note:$\lfloor x^n \rfloor$ Given a generating function A(x) we use$\lfloor x^n \rfloor$ A(x) to denote a_n, the coefficient of x ^{n}.()

Proposition

If we consider the following infinite matrix with 1's in the supper diagonal

$$A = \begin{pmatrix} u_0 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ u_1 & v_1 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ u_2 & v_2 & v_1 & 1 & 0 & 0 & 0 & . & . & . \\ u_3 & v_3 & v_2 & v_1 & 1 & 0 & 0 & . & . & . \\ u_4 & v_4 & v_3 & v_2 & v_1 & 1 & 0 & . & . & . \\ u_5 & v_5 & v_4 & v_3 & v_2 & v_1 & 1 & . & . & . \\ u_6 & v_6 & v_5 & v_4 & v_3 & v_2 & v_1 & . & . & . \\ u_7 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix} \quad (6)$$

Let U (x) = $\sum_{n=0}^{\infty} u_n x^n$ and V (x) = $\sum_{n=1}^{\infty} v_n x^n$ be the generating functions for the sequence

u_0, u_1, u_2 ….. And v1, v_2... respectively

If A (x) =$\dfrac{U(x)}{1+V(x)}$ =$\sum_{n=0}^{\infty} a_{n+1} x^n$ then a_n = (-1) $^{n-1}$ D_n and 1+xA (-x) is the generation function of D_n.

Proof

As we have seen in the above examples consider the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ v_1 x & x & 0 & 0 & 0 & 0 & . & . & . \\ v_2 x^2 & c_1 x^2 & x^2 & 0 & 0 & 0 & . & . & . \\ v_3 x^3 & c_2 x^3 & c_1 x^3 & x^3 & 0 & 0 & . & . & . \\ v_4 x^4 & c_3 x^4 & c_2 x^4 & c_1 x^4 & x^4 & 0 & . & . & . \\ v_5 x^5 & c_4 x^5 & c_3 x^5 & c_2 x^5 & c_1 x^5 & x^5 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 x \\ u_2 x^2 \\ u_3 x^3 \\ u_4 x^4 \\ u_5 x^5 \\ . \\ . \\ . \end{pmatrix} \quad (*)$$

Summing on both sides of (*) we get $A(x) + a_1 V(x) + a_2 x V(x) + a_3x^2V(x) + \ldots = U(x)$

$\Rightarrow A(x) + A(x) V(x) = U(x)$

$\Rightarrow A(x)(1+V(x)) = U(x) \Rightarrow A(x) = \frac{U(x)}{1+V(x)}$

Now from (6) we have $D_1 = u_0$, $D_2 = u_0v_1 - u_1 \Rightarrow u_1 = u_0v_1 - D_1$ and from (*) $v_1a_1 + a_2 = u_1$ & $a_1 = u_0 \Rightarrow v_1u_0 + a_2 = u_0v_1 - D_2 \Rightarrow a_2 = -D_2$ similarly $a_3 = D_3$ and continuing inductively and if x = 1 in (*) for the nxn case we have by Cramer's rule as we have seen above $a_n = (-1)^{n-1} D_n$ where D_n is the determinant of the nxn upper left corner sub matrix of (6).

i.e. $D_1 = a_1$, $D_2 = -a_2$, $D_3 = a_3$, $D_4 = -a_4 \ldots$

Let $A(x) = a_1 + a_2x + a_3x^2 + a_4x^3 + \ldots$

$\Rightarrow A(-x) = a_1 - a_2x + a_3x^2 - a_4x^3 + \ldots$

$\Rightarrow xA(-x) = a_1x - a_2x^2 + a_3x^3 - a_4x^4 + \ldots$

$\Rightarrow 1 + xA(-x) = 1 + a_1x - a_2x^2 + a_3x^3 - a_4x^4 + a_5x^5 + \ldots = 1 + D_1x + D_2x^2 + D_3x^3 + D_4x^4 + \ldots$

$\Rightarrow 1 + xA(-x)$ is the generating function for D_n, where $a_0 = D_0 = 1$

Example: If

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 & . & . & . \\ -1 & 2 & 0 & 2 & 1 & 0 & 0 & . & . & . \\ 1 & 0 & 2 & 0 & 2 & 1 & 0 & . & . & . \\ -1 & 2 & 0 & 2 & 0 & 2 & 1 & . & . & . \\ 1 & 0 & 2 & 0 & 2 & 0 & 2 & . & . & . \\ -1 & 2 & 0 & 2 & 0 & 2 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}. \quad \textbf{(1)}$$

Then

$$|1| = 1 = D_1, \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3 = D_2, \begin{vmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 7 = D_3, \begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ -1 & 2 & 0 & 2 \end{vmatrix} = 15 = D_4, \ldots \ldots \ldots$$

$D_{n+1} = 2D_n + 1$

Are the determinant of some nxn upper left corner sub matrices one way to determine say D_5, is as follows. Consider the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \textbf{(2)}$$

Where the right hand side is the first column from the original matrix of (1) for n=5

By Cramer's rule and if two columns are interchanged the determinant changes only sign. Then, we have

$$a_5 = \frac{\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 \\ 2 & 0 & 2 & 1 & -1 \\ 0 & 2 & 0 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 & 1 \end{vmatrix}} = 31 = D_5 \text{ in general by induction we have } a_n=(-1)^{n-1} D_n \quad \textbf{(3)}$$

Now let us introduce the generating functions for the columns of (1) and rewrite it as (2) we get the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 2x & x & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 2x^2 & x^2 & 0 & 0 & 0 & . & . & . \\ 2x^3 & 0 & 2x^3 & x^3 & 0 & 0 & . & . & . \\ 0 & 2x^4 & 0 & 2x^4 & x^4 & 0 & . & . & . \\ 2x^5 & 0 & 2x^5 & 0 & 2x^5 & x^5 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} 1 \\ -x \\ x^2 \\ -x^3 \\ x^4 \\ -x^5 \\ . \\ . \\ . \end{pmatrix} \quad \textbf{(4)}$$

Then the right hand side of (4) has a generating function $U(x) = 1-x+x^2-x^3+\ldots = \dfrac{1}{1+x}$

And except the elements on the main diagonal of the first matrix of left of (4), i.e excluding the1`s on the main diagonal of (4), the 1st column has a generating function

$V(x)=2x+2x^3+2x^5+\ldots=\dfrac{2x}{1-x^2}$

The 2nd column has a generating function xV(x)

The 3rd column has a generating function $x^2V(x)$

letting $A(x) = a_1+ a_2x + a_3x^2+$ as the generating function for the sequence $\langle a_1, a_2, a_3\ldots\rangle$ and summing on both sides of (4) we get

$$A(x) + a_1 \frac{2x}{1-x^2} + a_2 x \frac{2x}{1-x^2} + a_3x^3 \frac{2x}{1-x^2} +\ldots = \frac{1}{1+x}$$

$$\Rightarrow A(x) = \frac{1-x}{1+2x-x^2} = \frac{1-x^2}{1+2x-x^2}\left(\frac{1}{1+x}\right) = \frac{\frac{1}{1+x}}{1+\frac{2x}{1-x^2}} = \frac{U(x)}{1+V(x)}$$

* here $A(x)=a_1+a_2x+a_3x^2+a_4x^3+\ldots$

$\Rightarrow A(-x)=a_1-a_2x+a_3x^2-a_4x^3+\ldots$

$\Rightarrow xA(-x) = a_1x-a_2x^2+a_3x^3-a_4x^4+\ldots$

$$\Rightarrow A(-x) = \frac{1+x}{1-2x-x^2}$$

$$\Rightarrow xA(-x) = \frac{x+x^2}{1-2x-x^2}$$

$$\Rightarrow 1+xA(-x) = \frac{x+x^2+1-2x-x^2}{1-2x-x^2}$$

$$\Rightarrow 1+xA(-x) = \frac{1-x}{1-2x-x^2}$$

$$\Rightarrow 1+x A(-x)=1+a_1x-a_2x^2+a_3x^3-a_4x^4+\ldots=\frac{1-x}{1-2x-x^2}$$

$$=1+D_1x+D_2x^2+D_3x^3+D_4x^4+\ldots=\frac{1-x}{1-2x-x^2}$$

$\Rightarrow \dfrac{1-x}{1-2x-x^2}$ is the generating function for D_n where $a_0 = 1 = D_0$

Here $U(X) = \dfrac{1}{1+x}$ and $V(x) = \dfrac{2x}{1-x^2}$ in closed form.

Example: If

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 4 & 1 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ 9 & 1 & 1 & 1 & 0 & 0 & 0 & . & . & . \\ 16 & 1 & 1 & 1 & 1 & 0 & 0 & . & . & . \\ 25 & 1 & 1 & 1 & 1 & 1 & 0 & . & . & . \\ 36 & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . \\ 49 & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . \\ 81 & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}. \qquad \textbf{(1)}$$

Then

$$|1| = 1 = D_1, \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = -3 = D_2, \begin{vmatrix} 1 & 1 & 0 \\ 4 & 1 & 1 \\ 9 & 1 & 1 \end{vmatrix} = 5 = D_3, \begin{vmatrix} 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 \\ 9 & 1 & 1 & 1 \\ 16 & 1 & 1 & 1 \end{vmatrix} = -7 = D_4, \dots \dots \dots$$

$D_{n+1} = -D_n + 2(-1)^{n+2}$

Are the determinant of some nxn upper left corner sub matrices one way to determine say D_5, is as follows. Consider the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{pmatrix} \qquad \textbf{(2)}$$

Where the right hand side is the first column from the original matrix of (1) for n=5

By Cramer's rule and if two columns are interchanged the determinant changes only sign. Then, we have

$$a_5 = \frac{\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 4 \\ 1 & 1 & 1 & 0 & 9 \\ 1 & 1 & 1 & 1 & 16 \\ 1 & 1 & 1 & 1 & 25 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix}} = 9 = D_5$$ in general by induction we have $a_n = (-1)^{n-1} D_n$ **(3)**

Now let us introduce the generating functions for the columns of (1) and rewrite it as (2) we get the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ x & x & 0 & 0 & 0 & 0 & . & . & . \\ x^2 & x^2 & x^2 & 0 & 0 & 0 & . & . & . \\ x^3 & x^3 & x^3 & x^3 & 0 & 0 & . & . & . \\ x^4 & x^4 & x^4 & x^4 & x^4 & 0 & . & . & . \\ x^5 & x^5 & x^5 & x^5 & x^5 & x^5 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} 1 \\ 4x \\ 9x^2 \\ 16x^3 \\ 25x^4 \\ 36x^5 \\ . \\ . \\ . \end{pmatrix} \quad \textbf{(4)}$$

Then the right hand side of (4) has a generating function $U(x) = 1+4x+9x^2+16x^3+\ldots = \frac{1+x}{(1-x)^3}$

And except the elements on the main diagonal of the first matrix of left of (4),

i.e excluding the1`s on the main diagonal of the first matrix of left of (4), the 1st column has a generating function

$V(x)=x+x^2+x^3+\ldots = \frac{x}{1-x}$

The 2nd column has a generating function $xV(x) = x^2+x^3+\ldots$

The 3rd column has a generating function $x^2V(x)$

letting $A(x) = a_1 + a_2x + a_3x^2 +$ as the generating function for the sequence $\langle a_1, a_2, a_3 \ldots \rangle$ and summing on both sides of (4) we get

$$A(x) + a_1 \frac{x}{1-x} + a_2 x \frac{x}{1-x} + a_3 x^3 \frac{x}{1-x} + \ldots = \frac{1+x}{(1-x)^3}$$

$$\Rightarrow A(x) + (a_1 + a_2x + a_3x^2 + \cdots)\frac{x}{1-x} = \frac{1+x}{(1-x)^3}$$

$$\Rightarrow A(x)[1+\frac{x}{1-x}]=\frac{1+x}{(1-x)^3}$$

$$\Rightarrow A(x)[1+V(x)]=U(x)$$

$$\Rightarrow A(x)=\frac{U(x)}{1+V(x)}$$

*here $A(x)=a_1+a_2x+a_3x^2+a_4x^3+\ldots=\frac{1+x}{(1-x)^2}=\frac{1+x}{1-2x+x^2}$

$$\Rightarrow A(-x)=a_1-a_2x+a_3x^2-a_4x^3+\ldots=\frac{1-x}{1+2x+x^2}$$

$$\Rightarrow xA(-x)=a_1x-a_2x^2+a_3x^3-a_4x^4+\ldots=x(\frac{1-x}{1+2x+x^2})=\frac{x-x^2}{1+2x+x^2}$$

$$\Rightarrow 1+x A(-x)=1+a_1x-a_2x^2+a_3x^3-a_4x^4+\ldots=\frac{1+2x+x^2+x-x^2}{(1+x)^2}$$

$$=1+D_1x+D_2x^2+D_3x^3+D_4x^4+\ldots=\frac{1+2x+x^2+x-x^2}{(1+x)^2}=\frac{1+3x}{(1+x)^2}$$

$\Rightarrow 1+x A(-x)=1+a_1x-a_2x^2+a_3x^3-a_4x^4+\ldots=\frac{1+3x}{(1+x)^2}$ is the generating function for

D_n where $a_0=1=D_0$

Here $U(X)=\frac{1+x}{(1-x)^3}$ and $V(x)=\frac{x}{1-x}$ in closed form.

$$\frac{1+3x}{(1+x)^2}=\frac{1}{(1+x)^2}+\frac{3x}{(1+x)^2}=1+\sum_{n=1}^{\infty}(-1)^{n+1}(3n)x^n+\sum_{n=1}^{\infty}(-1)^{n+1}(-n-1)x^n$$

$$=1+\sum_{n=1}^{\infty}(-1)^{n+1}(2n-1)x^n$$

$\Rightarrow D_n=(-1)^{n+1}(2n-1)\ \forall n\geq 1$ And $D_0=1$

Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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REFERENCES

1. SEYOUM GETU, Evaluating Determinants via Generating functions, Howarg University Washington, Dc20059, Reprinted from the Mathimatics magazine Vol.64.No.1.February 1991.
2. Daniel L.A CoHen, Basic Techniques of combinatorial Theory.
3. David C.Lay, Linear Algebra and its Applications, 3 rd edition.
4. Purna Chandra Biswal (Ass.prof of mathematics National institute of science and Technology palur Hills, Berhampur), Discrete Mathematics and Graph theory, prentic Hall of India private limited, NewDelhi, 2005
5. Nch in lyengar, Discrete mathematics, Vikas publishing house pvt LTD, 2004
6. Amdeberhan Ayeligne, Seminal Report on Application of Generating functions, AAU 2007
7. Discrete Mathematics and Its Applications, Seventh Edition, Kenneth H. Rosen Monmouth University (and formerly AT&T Laboratories page 537